DIRECT AND INDIRECT MEASURES OF AREA

Within the General Mathematics course, students learn to make judgements about measurement errors. This contributes to their being able to demonstrate the outcome: "A student determines the degree of accuracy of measurements and calculations" (P7). In particular, students are expected to learn to determine the appropriate units to use when measuring physical attributes and to recognise that accuracy of physical measurement is limited to $\pm\frac{1}{2}$ of the smallest unit of which the measuring instrument is capable.

Any measurement produces some error. Consequently, determining the error in a measurement is a very practical application of mathematics. If I measure the three sides of a triangle to the nearest millimetre as, say, 31, 63 and 66 ($\pm$ 0.5 mm), and use Heron's formula to determine the area, what are sensible bounds for the area?

Using $s=\frac{a+b+c}{2}$ and $A=\sqrt{s(s-a)(s-b)(s-c)}$ with the specified side lengths produces a calculated area of A = 966 mm^2 (to nearest mm^2). If I accept a reasonable practical upper bound as being formed by having all of the measurements one millimetre more than my measurements and a lower bound as one millimetre less than my measurements, the area varies between 921 mm^2 and 1012 mm^2 (to nearest mm^2). Given the range, my indirect or calculated measure of area appears to be overly precise, as indeed are my error bounds.

We have made two departures from the General Mathematics syllabus. The first is the use of Heron's formula for calculating the area of a triangle. Students in the General Mathematics course are not required to use Heron's formula. Yet, if you are presented with a triangular shape and asked to determine its area, it is often easier to measure the three side lengths than to measure the perpendicular height. The second variation is that we have doubled the traditional error bounds. I think this is justified when we are measuring to the nearest millimetre. The other sources of error that we overlook in our calculations, such as parallax error or zero point error, can become quite significant at this level of accuracy.

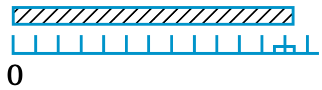


Fig. 1 Measurement error

From Figure 1, taking the accuracy of the physical measurement as limited to $\pm\frac{1}{2}$ of the smallest unit of which the measuring instrument is capable, assumes that any zero point error is random. That is, the distribution of the zero point error is just as likely to produce a low reading as a high reading and so will effectively "cancel out" error from this effect. This is likely to be the case with different people making the same measurement. However, with the same person making multiple measurements, I believe there is a greater chance of the zero point error being a systematic error rather than a random error.

Imagine a length that was very close to lying halfway between two consecutive millimetre marks. Presumably, the measure would be rounded down as many times as it was rounded up. The error bounds are only consistent with $\pm\frac{1}{2}$ of the smallest unit of which the measuring instrument is capable if we assume any zero point error to be random and not systematic.

No measurement of a continuous variable can yield one absolutely true value. Rather, the best we can hope for is a good estimate. Data collected in an experiment are frequently normally distributed around a mean value. As the normal distribution arises naturally in measurement, it is worthwhile laying the foundations for General Mathematics students' understanding of the distribution through practical measurement tasks.

If, instead of measuring the three sides, I measure the base and the perpendicular height, does this reduce my likely error? I appear to make only two measurements, so I should reduce my error. The third measurement that I make is, of course, the angle measure associated with the perpendicular. When students calculate the error involved in determining the area of a triangle, do they take this error into account? It is quite easy to determine the answer to this question by making sure that we include a practical component into our assessment of the measurement topic.

Practical assessment in measurement can be implemented in a range of ways. As a beginning, you could draw a triangle and have your students determine the area by measuring and calculating.

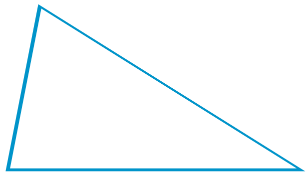


Fig. 2 Find the area of the above triangle.

If students are asked to record all of the measurements with appropriate precision, we have quite a simple yet challenging question. Finding the perpendicular height must rely on determining the perpendicular to within ± 0.5 degree. How much error is introduced in calculating the area of a triangle if the perpendicular height varies by ± 0.5 degree?

In the topic "Right-angled triangles (M4)", students are expected to use trigonometry to find the length of an unknown side. If we vary the "altitude" by ± 0.5 degree, as in Figure 3, the value of the altitude will change to $x' = \dfrac{x}{\sin 89.5°}$.

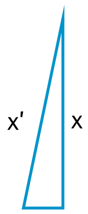


Fig. 3 The "altitude" varying by half a degree.

Owing to the symmetry in the problem, this will determine the maximum likely error in the altitude attributable to an inaccurate angle measurement. As a percentage error, this corresponds to approximately 0.004 %. This is a surprisingly small error. It does appear that using the traditional calculation of the area of a triangle as half the base times the perpendicular height will reduce the error in the calculated area!

Direct calculation

What happens if, instead of deriving the area, I calculate the area by counting squares on a grid overlay (Fig. 4)?

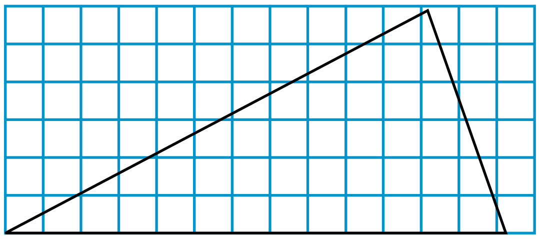


Fig. 4 Grid overlay

We could imagine using a millimetre grid overlay and counting only squares belonging to the triangle that were one half unit or greater. This method of directly measuring area has some interesting questions associated with it. For example, does it have any effect on the final answer which side of the triangle I align to the grid? As the error of measurement is associated with the number of partial squares formed by the triangle, this becomes quite a reasonable investigation in its own right.

Triangles with the same area, base and height

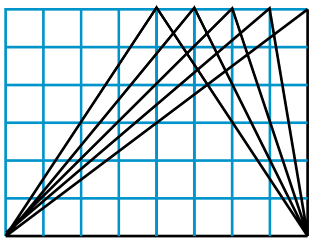


Fig. 5 Triangles with the same area

Each of the triangles in Figure 5 has an area of 24 units^2. Starting from the extreme right, the hypotenuse of the right-angled triangle passes through 12 squares. Moving to the next triangle, the short side passes through 6 squares and the long side passes through 12 squares. For the next triangle, the short side passes through 6 squares and the long side passes through 6 squares.

In general, the diagonal of an m x n rectangle passes through $m + n -$ hcf(m, n), where hcf(m, n) is the highest common factor of m and n. Considering the next triangle as made up of two rectangles we have:

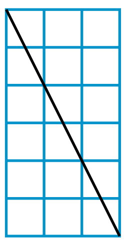


Fig. 6 3 + 6 – hcf(3, 6) = 6

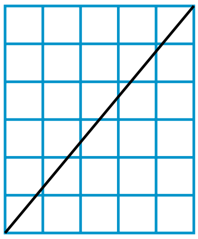


Fig. 7 5 + 6 – hcf(5, 6) = 10

That is, the two sides pass through 6 and 10 squares. We can see that, as the apex of the triangle varies, the number of partial squares formed will also vary. In practice this has little to no effect on the calculation of the area while our decisions are founded on half the smallest unit, that is half a square. If we can align the grid to the vertices of the triangle, so that we can use the area of the triangle as half of the enclosing rectangle, we minimise our error. This is an advantage of using a square as the unit of measure. Estimating half a linear unit is quite difficult. For example, try estimating the midpoint of the following interval.

Fig. 8 Interval PQ

Estimating half a square can be much simpler as it uses the properties of a square and isn't really an estimate. Recognising the following as half squares depends more on rotational symmetry than reflection. That is, it is generally not as easy as locating the diagonal.

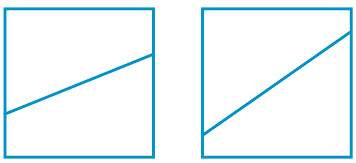


Fig. 9 Half squares

Estimating the error in direct measurement of area is not as easy as it first appears. It is a function of the smallest unit of measurement and the relationship between that unit and the dimensions of the triangle.

Precision

The introduction of powerful handheld calculators appears to have had a strange impact on the general perception of precision. Calculators that effortlessly produce calculations correct to eight decimal places or more encourage the idea that you can have whatever level of precision you would like at the press of a button. If you want a more accurate answer, don't you just buy a better calculator?

When Allen hex key wrenches have their sizes identified as 2.0 mm, 3.0 mm and 4.0 mm, is this just advertising? Although the answer may appear obvious to mathematics, science and technology teachers, many people would consider the final zeros as redundant. Didn't someone tell you that 2.0 and 2.00 were just the same as 2? In the world of pure mathematics they are the same but in the world of real measurement, they are different. The difference in precision between a measurement of 2 mm and one of 2.0 mm is shown in Figure 10.

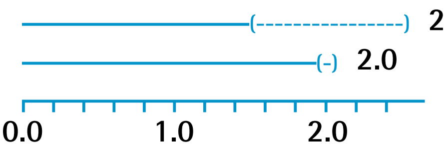


Fig. 10 Comparing measurements of 2 and 2.0

A measurement of 2 mm is between 1.5 mm and 2.5 mm, whereas a measurement of 2.0 mm indicates that the actual length is between 1.95 mm and 2.05 mm. The more precise a measurement is, the less variance is allowed.

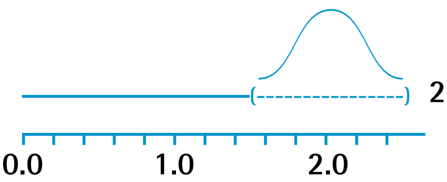


Fig. 11 Variance associated with a measurement of 2 mm.

If we turn the process around, we can ask questions such as the following:

> *A measurement and the associated error bounds are shown on the diagram. What was the measurement?*

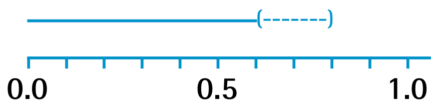


Fig. 12 Measurement and error

Although measurement is sometimes thought to use only basic mathematical concepts, this view overlooks the integration of mathematical thinking often required by measurement problems. Measurement tasks provide opportunities to draw together applications of algebra, geometry and trigonometry.

A measurement task

The following task is designed to assess students' understanding of error and its effect on calculated measurements.

ASSESSMENT TASK: Measuring the area of a triangle

Syllabus links

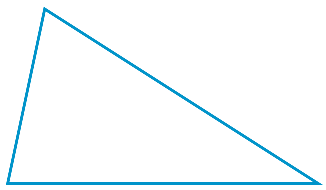
- Further applications of area and volume (M5)
- Applications of trigonometry (M6)

Outcomes addressed
A student:

- determines the degree of accuracy of measurements and calculations (P7)
- interprets the results of measurements and calculations and makes judgements about reasonableness (H7)
- integrates mathematical knowledge and skills from different content areas in exploring new situations (H2).

The task

1. Determine the area of the triangle below through measurement and calculation.



   Record all of the measurements you make and indicate any error band associated with your measurements and your final answer.
2. Finding the perpendicular height relies on determining the perpendicular to within ± 0.5 degree. How much error is introduced in calculating the area of a triangle if the perpendicular height varies by ± 0.5 degree?
3. The area of a triangle can also be found by measuring the three sides and using Heron's rule. To use Heron's rule, you need to find the semi-perimeter, s. To calculate s, add up the three side lengths (a, b and c) and halve the total $s = \frac{a + b + c}{2}$. The area of the triangle is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$. Work out the area of the triangle using Heron's rule and compare it to your initial calculation.
4. Compare the calculated error arising from the two different methods for finding the area of a triangle.
5. Determine the area of the triangle using $A = \frac{1}{2}ab\sin C$.
6. Which of the three methods appears to produce the most accurate answer and why?

Marking criteria
Your response will be assessed according to how well you:

- measure and record to appropriate levels of precision (3 marks)
- use a trigonometric ratio to determine a side length in a right triangle (2 marks)
- substitute into algebraic formulae and interpret the results (4 marks)
- determine errors in calculated area resulting from errors in measurement (4 marks)
- compare the error associated with different methods of calculating area (2 marks).

Packaging

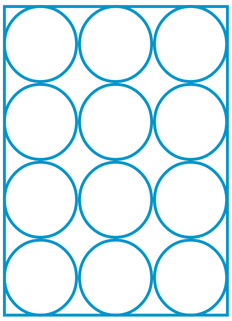
As packaging is a major component of landfill, the effective use of space in package design is important for both environmental and economic reasons.

In *Further applications of area and volume* (M5), students could design cost-effective packaging. There are many competing characteristics associated with soft-drink packaging. For example, soft-drink packaging can be designed so that it uses shelf space well, uses the least packaging material or uses the cheapest packaging material. Soft-drink cans might also be packaged to be cooled efficiently.

The primary packaging of soft drink is the can. The container used to hold several cans is secondary packaging. The design of secondary packaging can form the basis of a lesson using mathematical modelling.

What are the dimensions of a standard soft drink can and what is the capacity of the can? Soft drink cans are often supplied in a twelve-pack, in an arrangement as shown below.

If the radius of each can is approximately 3.2 cm,

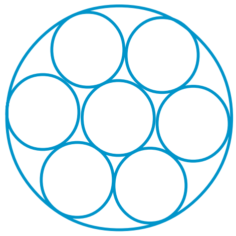


what percentage of the package space is used by the cans? Repeat your calculation, but this time use r to represent the radius and show that your answer is independent of the size of the cans. If the height of each can is about 12 centimetres, show that the per cent of space used is independent of the height.

Determine the area of packaging material used by the twelve-pack. To simplify the calculation, assume no overlap in the packaging. Would a six-pack in a rectangular design use more or less packaging per can? What happens to the amount of packaging per can if an eighteen-pack is used?

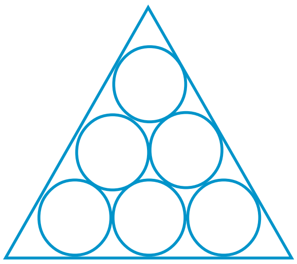
Package shape

We have used a rectangular arrangement as the basis of the packages that we have investigated so far. Below is an outline for a cylindrical seven-pack.



Determine the per cent of package space used by the cans. Compare this design to the standard twelve-pack.

A six-pack could also be arranged in a triangular formation as below.



Determine the per cent of package space used by the cans. By using Heron's rule, show that the area of an equilateral triangle is given by $\frac{\sqrt{3}\,a^2}{4}$ where a is the side length of the triangle. Compare this design to the standard twelve-pack.

Design your own package. Try to find a shape for which the per cent of space used by the cans is greater than in any of the designs considered so far.

Students who are not confident at carrying out a modelling task will benefit from additional structure being provided in class. Greater access to the task can be provided through a focus on the practical aspects of measurement. Use drink cans to obtain the measurements and have students create circular cardboard disks to act as models of the cans by using the cans as templates. Students can draw around the base of a drink can and cut out the resulting circle. Using cardboard circles in different arrangements allows students to create a practical approach rather than a theoretical solution to the problem.

USING EXTENDED INVESTIGATIONS IN STAGE 6 ASSESSMENT

Some schools have started to explore the use of longer assessment tasks in mathematics. One way to do this is to use an extended investigation. With an extended investigation, as with any assessment task, students need to be given advice as to what is expected of them. Following is the kernel of an extended investigation relating to determining minimum surface area within the Mathematics Extension 1 course.

Investigating maxima and minima in Mathematics Extension 1

GENERAL ADVICE FOR STUDENTS

In this investigation, you are required to examine a problem, or series of related problems, involving the determination of maximum or minimum values of a function, or local maximum or minimum values with respect to one of the variables. You need to give careful consideration to domain limitations and other factors that might relate to the reasonableness of your solution. Following this introductory task, a more general version of the problem will be presented which requires a greater depth of analysis.

UNDER WRAPS

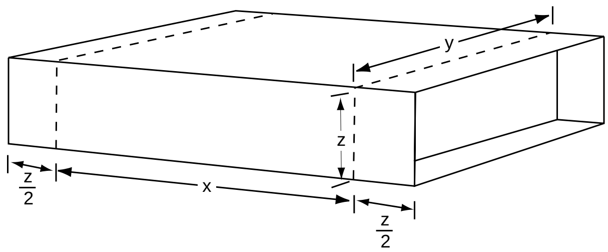
There are many situations in which we need to minimise the surface area of a solid. At times we may be concerned with the area of material required to "wrap" a solid. Assume that the solid under consideration in this investigation has a **fixed** volume. You will examine the relationships between the volume, total surface area and dimensions of a solid to determine the conditions necessary for a solid, or its wrapping, to have a minimum surface area.

(a) (i) Consider a closed rectangular box of fixed volume and dimensions x, y, z. To begin with, assume that the value of z is fixed while x and y may vary. You may use numerical values for z and for the volume, or work with general arbitrary constants. Show that the surface area of the box is a minimum when $x = y$.

(ii) Now let the value of z vary as well, keeping $x = y$. Determine the shape of the box with the minimum surface area. How can you be sure your answer gives a minimum?

(iii) Return to the general box of fixed volume and dimensions x, y, z. Suppose that the top of the rectangular box is removed so that it has five sides instead of six. Using a similar method to the approach in parts (i) and (ii), find the shape of an open box (of a particular volume) with minimum surface area..

(b) (i) Now consider the problem of wrapping a box with dimensions x, y, z with paper, as shown in the following figure. Assume $z \leq y$.



Also assume that the paper to be used is just sufficient to wrap the box, folding the ends in a conventional way. Given that the volume of the box is fixed, find the dimensions of the box for which the area of wrapping paper is a minimum.

(ii) It would be more realistic if the wrapping paper overlapped itself in some way. Would your result be any different if you took into account the overlap? Explain the reasoning behind the answer you give.

This assessment task is similar to investigations used with the senior assessment tasks in Victoria.

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